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Chebyshev's Inequality: An Improvement

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ABSTRACT: The classic Chebyshev's inequality P{$|X-\mu| < k\sigma|$} does not work when $k < 1$, where μ and σ are mean and standard deviation of the distribution respectively. This paper has addressed this problem of Chebyshev's inequality and proposed an improvement of the classic Chebyshev's inequality. It has been shown that P{$|X-\mu| < k\sigma$} ≤ $2kf(m)\sigma$ (where m is mode and *f*() is *pdf* (Probability Density Function) or P{$|X-\mu| < k\sigma$} ≤ $2k\sigma$ (if *m* is not known). The improvement has been verified with two major distributions: *Normal* and *Gamma* distribution, and it has been found that the improved inequality conforms to those major distributions.

KEYWORDS: Chebyshev inequality, Gamma distribution, Normal distribution.

1. INTRODUCTION

Probability theory [1] is full of different inequalities. The classic Chebyshev's inequality ([2], [3]) is one of them. It is perhaps one of the earliest, most common and highly used inequalities in probability theory. It can be stated in different forms. Lots of research ([4], [5], [6]) on Chebyshev's inequality have been done by different researchers. In simplest form, it states that for a random variable *X*, P{$|X-\mu| \le k\sigma$} ≥ $1 - 1/k^2$ and P{$|X-\mu| \ge k\sigma$} ≤ $1/k^2$, where μ and σ are mean and standard deviation respectively of the random variable *X* and $k > 1$. This inequality is good in the sense that it does not require other parameters like mean, median, mode, etc. of the distribution. This is the major advantage of the inequality. However, the inequality won't work properly when $k < 1$ and this is the major disadvantage of the inequality. This paper has addressed this problem and presented an improved version of the Chebyshev's inequality for $k < 1$. The next section gives the proof of the Chebyshev's inequality followed by section 3, which presents the proposed improvement. Finally, section 4 gives the experimental results of the proposed inequality.

2. CHEBYSHEV'S INEQUALITY

Let μ and σ be mean and standard deviation respectively of the distribution of a random variable *X* and $k > 1$. Then, Chebyshev's Inequality (*Figure* 1) states that P{$|X-\mu| \le k\sigma$} ≥ $1 - 1/k^2$ and P{$|X-\mu| \ge k\sigma$} ≤ $1/k^2$. As stated above, the main advantage of the inequality is that it needs only the value of *k*, nothing else. As for example, let $X \sim N(1, 2)$ and $k = 3$.

Then {P $|X-\mu| \le k\sigma$} = 0.9973 and by Chebyshev's inequality {P$|X-\mu| \le k\sigma$} >= 0.8888, which is correct. The inequality works well for values of $k > 1$, but it fails when $k < 1$. For the same example, if $k = 0.5$, by Chebyshev's inequality {P$|X-\mu| \le k\sigma$} ≥ −3. This is mathematically correct, but not in true sense. This is the main disadvantage of the inequality. The next section gives the proposed improvement of the Chebyshev's inequality, which tries to overcome this disadvantage of Chebyshev's inequality.

Let *f, μ and σ* be pdf, mean and standard deviation respectively of a random variable *X*. Then

$$\sigma^2 = \int_{-\infty}^{\infty} (X-\mu)^2 f(X)$$

=> $\sigma^2 = \int_{-\infty}^{\mu-k\sigma} (X-\mu)^2 f(X) + \int_{\mu-k\sigma}^{\mu+k\sigma} (X-\mu)^2 f(X) + \int_{\mu+k\sigma}^{\infty} (X-\mu)^2 f(X)$

=> $\sigma^2 \ge \int_{-\infty}^{\mu-k\sigma} (X-\mu)^2 f(X) + \int_{\mu+k\sigma}^{\infty} (X-\mu)^2 f(X) \ \left(\because \int_{\mu-k\sigma}^{\mu+k\sigma} (X-\mu)^2 f(X) \ge 0\right)$

=> $\sigma^2 \ge k^2\sigma^2 \int_{-\infty}^{\mu-k\sigma} f(X) + k^2\sigma^2 \int_{\mu+k\sigma}^{\infty} f(X)$

=> $\sigma^2 \ge k^2\sigma^2 \left(\int_{-\infty}^{\mu-k\sigma} f(X) + \int_{\mu+k\sigma}^{\infty} f(X)\right)$

=> $\sigma^2 \ge k^2\sigma^2 \{ P(|X-\mu| \ge k\sigma)\}$

=> $P(|X-\mu| \ge k\sigma) \le \frac{1}{k^2}$ *and* $P(|X-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2}$ $(k > 1)$

Figure 1: Chebyshev's Inequality

3. THE PROPOSED IMPROVEMENT

Let μ, σ, *m* and f() be mean, standard viation, mode and pdf of a random variable *X*. Then, the proposed improvement states that $P\{|X-\mu| \leq k\sigma\} \leq 2k\, f(m)\sigma$ (if *m* is known) and $P\{|X-\mu| \leq k\sigma\} \leq 2k\sigma$ (if *m* is not known).

Proof:

$$P\{|X-\mu| \leq k\sigma\} = \int_{\mu-k\sigma}^{\mu+k\sigma} f(X)dx$$

$$P\{|X-\mu| \leq k\sigma\} < \int_{\mu-k\sigma}^{\mu+k\sigma} f(m)dx \ (\because f(X) < f(m))$$

$$P\{|X-\mu| \leq k\sigma\} < f(m)\int_{\mu-k\sigma}^{\mu+k\sigma} dx$$

$$P\{|X-\mu| \leq k\sigma\} < f(m)\ [x]_{\mu-k\sigma}^{\mu+k\sigma}$$

$$P\{|X-\mu| \leq k\sigma\} < 2f(m)k\sigma$$

Figure 2: The Proposed Improvement

For useful results, the term 2*f(m)k*σ should be less than 1 i.e. $k < \frac{1}{2f(m)\sigma}$ or $\sigma < \frac{1}{2f(m)k}$. This is obvious, because for large σ, variable values will be more scattered and away from the mean value μ.

4. EXPERIMENTAL RESULTS

The proposed improvement has been tested with Gamma distribution (*Figure* 3) and Standard Normal distribution (*Figure* 4) for different values of k < 1. *Figure* 3 shows the result for three different Gamma distributions (for mean λ = 2, 3 and 4). In the figure, the "*Actual"* column gives the actual probabilities $P\{|X-\lambda| \leq k\sqrt{\lambda}\}$ (standard deviation for Gamma distribution is $\sqrt{\lambda}$) and the columns "*Maximum*" gives the corresponding values of $2kf(\lambda-1)\sqrt{\lambda}$, where *f* and (λ -1) are the *pdf* and mode of the gamma distribution respectively. *Figure* 4 gives the experimental result of Standard Normal distribution. Here also, the column *"Actual Probability"* gives the $P\{|X| \leq k\}$ ($\because \mu = 0, \sigma = 1$ and mode = 0 for Standard Normal distribution)
and the column "*Maximum*" gives the corresponding values of 2*kf*(0), where *f*(0) = 0.3989 for Standard Normal distribution.

	λ =2		λ=3		λ=4	
K	Actual	Maximum	Actual	Maximum	Actual	Maximum
0.01	0.00765572	0.01046518	0.007760993	0.009976613	0.007814607	0.00896
0.02	0.015311439	0.020930361	0.015521727	0.019953225	0.015628824	0.01792
0.03	0.022967159	0.031395541	0.023281943	0.029929838	0.232650772	0.2688
0.04	0.030622876	0.041860721	0.031041382	0.039906451	0.308289808	0.3584
0.05	0.038278591	0.052325902	0.038799784	0.049883063	0.382205973	0.448
0.06	0.045934298	0.062791082	0.046556887	0.059859676	0.453871934	0.5376
0.07	0.053589995	0.073256263	0.054312429	0.069836289	0.522710626	0.6272
0.08	0.061245674	0.083721443	0.062066148	0.079812901	0.588100502	0.7168
0.09	0.068901327	0.094186623	0.069817776	0.089789514	0.649389535	0.8064
0.1	0.076556942	0.104651804	0.077567048	0.099766127	0.078081474	0.0896
0.2	0.153106197	0.209303607	0.154869592	0.199532253	0.155768338	0.1792
0.3	0.229609047	0.313955411	0.231620082	0.29929838	0.240278191	0.27776
0.4	0.305963465	0.418607214	0.30748563	0.399064506	0.315767264	0.36736
0.5	0.381969329	0.523259018	0.382067331	0.498830633	0.389482965	0.45696
0.6	0.457289756	0.627910822	0.454882333	0.598596759	0.460892968	0.54656
0.7	0.531407103	0.732562625	0.525350184	0.698362886	0.529415392	0.63616
0.8	0.603571996	0.837214429	0.592785555	0.798129012	0.594424854	0.72576
0.9	0.672743562	0.941866233	0.656400249	0.897895139	0.655267434	0.81536

Figure 3: Experimental Result (Gamma Distribution)

k	Actual Probability	Maximum
0.01	0.008	0.0080
0.02	0.016	0.0160
0.03	0.024	0.0239
0.04	0.032	0.0319
0.05	0.0398	0.0399
0.06	0.478	0.0479
0.07	0.0558	0.0559
0.08	0.0638	0.0638
0.09	0.0718	0.0718
0.1	0.0796	0.0798
0.2	0.1586	0.1596
0.3	0.2358	0.2394
0.4	0.3108	0.3192
0.5	0.383	0.3989
0.6	0.4514	0.4787
0.7	0.516	0.5585
0.8	0.5762	0.6383
0.9	0.6318	0.7181

Figure 4: Experimental Result (Standard Normal Distribution)

5. CONCLUSION

This paper has presented an improvement of the classic Chebyshev's inequality, which works for $k < 1$. The paper has also presented some experimental results for Normal and Gamma distribution. However, some more experimental results for other distributions are required to enhance the robustness of the improvement.

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